

LEFSCHETZ FIXED POINT THEOREMS FOR FOURIER-MUKAI FUNCTORS AND DG ALGEBRAS

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ABSTRACT. We propose some variants of Lefschetz fixed point theorem for Fourier-Mukai functors on a smooth projective algebraic variety. Independently we also suggest a similar theorem for endo-functors on the category of perfect modules over a smooth and proper DG algebra.

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1. INTRODUCTION

Leschetz fixed point theorem (LFP) is a principle that has many incarnations. One of its simplest forms is the following: Let $f : X \rightarrow X$ be a nice self-map of a nice space X . Then the number of fixed points of f equals the supertrace

$$\sum_i (-1)^i \operatorname{Tr} H_i(f)$$

where $H_i(f) : H_i(X) \rightarrow H_i(X)$ is the induced map on homology. The "number of fixed points" should be properly defined as the intersection $\Gamma(f) \cdot \Delta$ of the graph of f with the diagonal $\Delta \subset X \times X$.

In this paper we prove several variants of LFP theorem.

In the first part we work in algebraic-geometric setting. Let X be a smooth projective variety over a field k . Denote by $D^b(X)$ the bounded derived category of coherent sheaves on X . Let Y be another smooth projective variety over k and $E \in D^b(X \times Y)$. Then there is the corresponding Fourier-Mukai functor $\Phi_E : D^b(X) \rightarrow D^b(Y)$

$$\Phi_E(-) = \mathbf{R}p_*(E \otimes^{\mathbf{L}} q^*(-)),$$

where the maps p, q are the projections

$$X \xleftarrow{q} X \times Y \xrightarrow{p} Y.$$

One has a finite dimensional graded vector space $HH_\bullet(X)$ - the Hochschild homology of X . The functor Φ_E induces the linear map of graded spaces $HH_\bullet(\Phi_E) : HH_\bullet(X) \rightarrow HH_\bullet(Y)$. In particular if $X = Y$ we get endomorphisms $HH_i(\Phi_E) : HH_i(X) \rightarrow HH_i(X)$ for each $i \in \mathbb{Z}$. It is natural to define for $E \in D^b(X \times X)$ the "number of fixed points of Φ_E " to be the "intersection" of E with the diagonal $\Delta \subset X \times X$, i.e. as $\sum_i (-1)^i \dim HH_i(E)$ (see Definition 3.7). The following Hochschild homology version of LFP theorem (= Theorem 3.9) is easy to prove.

Theorem 1.1. *In the above notation there is the equality*

$$\sum_i (-1)^i \dim HH_i(E) = \sum_j (-1)^j \operatorname{Tr} HH_j(\Phi_E).$$

The proof of this theorem has a tautological flavor once basic functorial properties of $HH(X)$ have been established. Here the main references are [Cal1], [Ram], [MaSte].

It is as easy to prove the Hirzebruch-Riemann-Roch theorem for Hochschild homology (Proposition 3.12).

Assume now that $k = \mathbb{C}$. Then one can consider the singular cohomology $H^\bullet(X, \mathbb{C})$. Again an object $E \in D^b(X \times Y)$ induces the linear map $H^\bullet(\Phi_E) : H^\bullet(X, \mathbb{C}) \rightarrow H^\bullet(Y, \mathbb{C})$ which is the convolution with the cohomology class $ch(E) \cup \sqrt{td_{X \times Y}} \in H^\bullet(X \times Y, \mathbb{C})$ (here $ch(E)$ is the Chern character of E and $td_{X \times Y}$ is the Todd class of $X \times Y$). This map preserves the parity of the degree of cohomology, hence it is the sum of two linear operators $H^{ev}(\Phi_E) \oplus H^{odd}(\Phi_E)$. Next is the singular cohomology version of LFP theorem (= Theorem 4.3).

Theorem 1.2. *Let X be a complex smooth projective variety and let $E \in D^b(X \times X)$. Then there is the equality*

$$\sum_i (-1)^i \dim HH_i(E) = \text{Tr } H^{ev}(\Phi_E) - \text{Tr } H^{odd}(\Phi_E).$$

This theorem follows from Theorem 1.1 above and Theorem 1.2 in [MaSte] (which in turn is heavily based on [Cal1],[Cal2] and [Ram]).

Denote by $\Delta : X \rightarrow X \times X$ the diagonal embedding. It follows from the Grothendieck-Riemann-Roch theorem that

$$\sum_i (-1)^i \dim HH_i(E) = \int_{X \times X} ch E \cup \Delta_*(td_X) = \int_X \Delta^*(ch E) \cup td_X$$

(Remark 4.5). So the left hand side in Theorems 1.1,1.2,1.3 can be computed using the Chern character of E .

Consider now the singular homology $H_\bullet(X) = H_\bullet(X, \mathbb{C})$. Let $f : X \rightarrow X$ be a morphism. For each j we get the corresponding linear map $H_j(f) : H_j(X) \rightarrow H_j(X)$. Again it is natural to define the "number of fixed points of f " as the alternating sum $\sum_i (-1)^i HH_i(\mathcal{O}_{\Gamma(f)})$, where $\mathcal{O}_{\Gamma(f)} \in D^b(X \times X)$ is the structure sheaf of the graph $\Gamma(f)$ of the morphism f .

Theorem 1.3. *Let X be a smooth complex projective variety and let $f : X \rightarrow X$ be a morphism. Then in the above notation there is the equality*

$$\sum_i (-1)^i \dim HH_i(\mathcal{O}_{\Gamma(f)}) = \sum_j (-1)^j \text{Tr } H_j(f).$$

This theorem (= Theorem 5.1) is a consequence of the special case of Theorem 1.2 (when $E = \mathcal{O}_{\Gamma(f)}$) and the Poincare duality between the singular homology and cohomology of X . Theorem 1.3 is not new: a similar formula can be proved for any Weil (co)homology theory (of which singular cohomology is an example) (see for instance [Mus]). Nevertheless we consider it natural to derive Theorem 1.3 from Theorem 1.1, since in that last theorem both sides of the equality have the same nature - Hochschild homology.

Finally in Section 6 we prove yet another version of LFP theorem for two maps between different spaces (Theorem 6.1).

In the second part of the paper we want to consider a LFP theorem of categorical nature: the space X is a triangulated category T and the map f is an endofunctor $F : T \rightarrow T$. More precisely, let A be a smooth and proper DG algebra (over a fixed field k). A perfect DG bimodule $M \in \text{Perf}(A^{\text{op}} \otimes A)$ defines the endofunctor

$$\Phi_M = - \otimes_A^{\mathbf{L}} M : \text{Perf } A \rightarrow \text{Perf } A$$

where $\text{Perf } A$ is the triangulated category of perfect DG A -modules. It is natural to define the "number of fixed points of Φ_M " as the alternating sum $\sum_i (-1)^i \dim HH_i(M)$, where $HH_i(M)$ is the i -th Hochschild homology space of the DG bimodule M .

The functor Φ_M defines the endomorphism $HH_j(\Phi_M)$ of the Hochschild homology $HH_j(A)$ for each $j \in \mathbb{Z}$. We prove the following LFP theorem (= Theorem 8.2) for Φ_M (our assumption on A guarantees that all spaces involved have finite dimension).

Theorem 1.4. *Let A be a smooth and proper DG algebra over a field k and let $M \in \text{Perf } A^{\text{op}} \otimes A$ be a perfect DG bimodule. Then there is an equality of the two elements of k*

$$\sum_i (-1)^i \dim HH_i(M) = \sum_j (-1)^j \text{Tr } HH_j(\Phi_M).$$

Actually a proof of this theorem (but not the statement) is essentially contained in a beautiful preprint [Shk] of D. Shklyarov, where the Hirzebruch-Riemann-Roch (HRR) theorem for DG algebras is discussed. It turns out that the proofs of theorems HRR and LFP have much in common. Since the paper [Shk] seems to be unfortunately unpublished we thought it worthwhile to give a simultaneous presentation of theorems HRR and LFP. Thus most of what is contained in Part II is from [Shk].

This paper has two parts. It was our initial plan to deduce Theorem 1.1 from Theorem 1.4 using the description of the derived category $D^b(X)$ as the category $\text{Perf } A$ of perfect modules over a smooth and proper DG algebra A . But then we found a short self-contained proof of Theorem 1.1, so the two parts of this paper are completely independent (but parallel).

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Part 1. Lefschetz fixed point theorem for Fourier-Mukai functors**2. FOURIER-MUKAI FUNCTORS**

We fix a field k . All our varieties will be k -varieties.

If Z is a smooth projective variety we denote by $D^b(Z) = D^b(\text{coh}Z)$ the bounded derived category of coherent sheaves on Z .

Let X and Y be smooth and projective varieties over k . An object $E \in D^b(X \times Y)$ defines the corresponding *Fourier-Mukai* functor $\Phi_E : D^b(X) \rightarrow D^b(Y)$ by the formula

$$\Phi_E(-) = \mathbf{R}p_*(E \overset{\mathbf{L}}{\otimes} q^*(-)),$$

where p and q are the projections

$$X \xleftarrow{q} X \times Y \xrightarrow{p} Y$$

Denote by $\Delta : X \rightarrow X \times X$ the diagonal morphism. The object $\Delta_*\mathcal{O}_X \in D^b(X \times X)$ induces the identity functor $\text{id} = \Phi_{\Delta_*\mathcal{O}_X} : D^b(X) \rightarrow D^b(X)$.

Given another smooth projective variety Z and $E' \in D^b(Y \times Z)$ the composition of functors $\Phi_{E'} \cdot \Phi_E$ is isomorphic to the functor $\Phi_{E' * E}$, where $E' * E \in D^b(X \times Z)$ is the usual convolution of E' and E [Mu].

The functor Φ_E induces the linear map $H(\Phi_E)$ between $H(X)$ and $H(Y)$, where $H(-)$ denotes the Hochschild homology or the singular cohomology (if $k = \mathbb{C}$). We are going to prove LFP type theorems for these operators $H(\Phi_E)$. Later in Section 5 we prove an analogous theorem for singular homology.

3. HOCHSCHILD HOMOLOGY OF SMOOTH PROJECTIVE VARIETIES

Let X be a smooth projective variety of dimension n . We recall one of the many possible (equivalent) definitions of the Hochschild homology of X following [Cal1]. Namely let $S_X^{-1} = \omega_X^*[-n] \in D^b(X)$ denote the shift of the dual of the canonical line bundle of X . Consider the diagonal embedding $\Delta : X \rightarrow X \times X$. Then one defines

$$HH_i(X) = \text{Hom}_{D^b(X \times X)}^{-i}(\Delta_*S_X^{-1}, \Delta_*\mathcal{O}_X).$$

We put

$$HH(X) = HH_\bullet(X) = \bigoplus_i HH_i(X).$$

Alternatively

$$HH_i(X) = \text{Hom}_{D^b(X)}^{-i}(\mathcal{O}_X, \mathbf{L}\Delta^*\Delta_*\mathcal{O}_X).$$

Actually we will never need to use the definition of $HH(X)$ but rather some of its properties which we now summarize.

Properties of $HH(X)$.

1. $\dim HH(X) < \infty$ and $HH(\text{pt}) = HH_0(\text{pt}) = k$.
2. An object $E \in D^b(X \times Y)$ defines a degree preserving linear map $HH(\Phi_E) : HH(X) \rightarrow HH(Y)$. In particular if $f : X \rightarrow Y$ is a morphism, then the structure sheaf of its graph $\mathcal{O}_{\Gamma(f)}$ considered as an object of $D^b(X \times Y)$ or $D^b(Y \times X)$ defines the corresponding linear maps which we denote

$$f_* : HH(X) \rightarrow HH(Y), \quad f^* : HH(Y) \rightarrow HH(X).$$

The linear map $HH(\Phi_{\Delta_* \mathcal{O}_X})$ defined by the object $\Delta_* \mathcal{O}_X \in D^b(X \times X)$ is the identity.

3. The correspondence $E \mapsto HH(\Phi_E)$ is functorial: Given $E' \in D^b(Y \times Z)$ the convolution $E' * E \in D^b(X \times Z)$ defines the map $HH(\Phi_{E' * E})$ which is the composition $HH(\Phi_{E'}) \cdot HH(\Phi_E)$.
4. There exists the canonical *Kunneth isomorphism*

$$K : HH(X) \otimes HH(Y) \longrightarrow HH(X \times Y)$$

5. If $\sigma : X \times X \rightarrow X \times X$ denotes the transposition then the induced map

$$K^{-1} \cdot \sigma_* \cdot K : HH(X) \otimes HH(X) \rightarrow HH(X) \otimes HH(X)$$

is $a \otimes b \mapsto (-1)^{\deg(a) \deg(b)} b \otimes a$.

6. Let $E \in D^b(X \times Y)$ and $E' \in D^b(Z \times W)$. Then the following diagram commutes

$$\begin{array}{ccc} HH(X) \otimes HH(Z) & \xrightarrow{K} & HH(X \times Z) \\ HH(\Phi_E) \downarrow & \downarrow HH(\Phi_{E'}) & \downarrow HH(\Phi_{E \boxtimes E'}) \\ HH(Y) \otimes HH(W) & \xrightarrow{K} & HH(Y \times W) \end{array}$$

7. *The Euler class.* Consider an object $N \in D^b(X)$ as an object in $D^b(\text{pt} \times X)$. Define the Euler class of N as $Eu(N) = HH(\Phi_N)(1) \in HH_0(X)$. The map Eu descends to a group homomorphism

$$Eu : K_0(D^b(X)) \rightarrow HH_0(X).$$

Given $E \in D^b(X \times Y)$ the following diagram commutes

$$\begin{array}{ccc} D^b(X) & \xrightarrow{\Phi_E} & D^b(Y) \\ \downarrow Eu & & \downarrow Eu \\ HH(X) & \xrightarrow{HH(\Phi_E)} & HH(Y) \end{array}$$

If $X = \text{pt}$ and hence N is just a complex of vector spaces then

$$Eu(N) = \sum_i (-1)^i \dim H^i(N) \in HH(\text{pt}) = k.$$

Property 1 is clear; 2,3,7 are from [Cal1] (the Euler class is called the Chern character in [Cal1]) and 4,6 are from [MaSte]. The property 5 follows from the usual supercommutativity of the tensor product of complexes.

Pairing on HH . We want to consider the following pairing on $HH(X)$.

Definition 3.1. *Consider the diagram of morphisms*

$$X \times X \xleftarrow{\Delta} X \xrightarrow{p} \text{pt}$$

and define the map $\langle, \rangle_X : HH(X) \otimes HH(X) \rightarrow k$ as the composition

$$HH(X) \otimes HH(X) \xrightarrow{K} HH(X \times X) \xrightarrow{HH(\Delta^*)} HH(X) \xrightarrow{HH(p_*)} HH(\text{pt}) = k.$$

Remark 3.2. *It follows from Property 5 above that $\langle a, b \rangle_X = (-1)^{\deg(a) \deg(b)} \langle b, a \rangle_X$.*

Remark 3.3. *Apparently this pairing is not the same as the Mukai pairing considered by Caldararu [Cal1], although the two are closely related (see [Ram2]). Our pairing is a direct analogue of the pairing 7.12 below and the next lemma (and its proof) is similar to Lemma 8.5*

Notation. Given smooth projective varieties X, Y, Z, W and objects $E \in D^b(X \times Y), E' \in D^b(Z \times W)$ we denote the functor $\Phi_{E \boxtimes E'} : D^b(X \times Z) \rightarrow D^b(Y \times W)$ by $\Phi_E \boxtimes \Phi_{E'}$.

Let X and Y be smooth projective varieties and let $E \in D^b(X \otimes Y)$. Define

$$Eu(E)' \in HH(X) \otimes HH(Y)$$

to be the inverse image of $Eu(E)$ under the Kunneth isomorphism.

Lemma 3.4. *In the above notation the linear map $HH(\Phi_E) : HH(X) \rightarrow HH(Y)$ is the convolution with the class $Eu'(E)$. I.e. it is equal to the composition*

$$HH(X) \xrightarrow{\text{id} \otimes Eu'(E)} HH(X) \otimes HH(X) \otimes HH(Y) \xrightarrow{\langle, \rangle_X \otimes \text{id}} HH(Y).$$

Proof. First notice that the Fourier-Mukai functor $\Phi_E : D^b(X) \rightarrow D^b(Y)$ is isomorphic to the following composition of functors

$$\begin{array}{ccccccc} D^b(X \times \text{pt}) & \xrightarrow{\text{id} \boxtimes \Phi_E} & D^b(X \times X \times Y) & \xrightarrow{\Delta^* \boxtimes \text{id}} & D^b(X \times Y) & \xrightarrow{p_* \boxtimes \text{id}} & D^b(\text{pt} \times Y) \\ \parallel & & & & & & \parallel \\ D^b(X) & & & & & & D^b(Y) \end{array}$$

Now it follows from properties 3,4,6,7 above that the corresponding linear map $HH(\Phi_E) : HH(X) \rightarrow HH(Y)$ is equal to the following composition

$$\begin{array}{ccccc}
HH(X) & & & & \\
\parallel & & & & \\
HH(X) \otimes HH(\text{pt}) & \xrightarrow{\text{id} \otimes Eu(E)} & HH(X) \otimes HH(X \times Y) & & HH(Y) \\
& & \downarrow \text{id} \otimes K^{-1} & & \parallel \\
& & HH(X) \otimes HH(X) \otimes HH(Y) & & HH(\text{pt}) \otimes HH(Y) \\
& & \downarrow K \otimes \text{id} & & \uparrow HH(p_*) \otimes \text{id} \\
& & HH(X \times X) \otimes HH(Y) & \xrightarrow{HH(\Delta^*) \otimes \text{id}} & HH(X) \otimes HH(Y)
\end{array}$$

Finally notice that the composition $\text{id} \otimes (K^{-1} \cdot Eu(E))$ is equal to $\text{id} \otimes Eu'(E)$. Also the composition $(HH(p_*) \cdot HH(\Delta^*) \cdot K) \otimes \text{id}$ is the map $\langle, \rangle_X \otimes \text{id}$. This proves the lemma. \square

Corollary 3.5. *The pairing \langle, \rangle_X is nondegenerate.*

Proof. The Fourier-Mukai functor $\Phi_{\Delta_* \mathcal{O}_X} : D^b(X) \rightarrow D^b(X)$ is isomorphic to the identity. Hence it follows from Lemma 3.4 that the map $HH(X) \rightarrow HH(X)^*$ defined by \langle, \rangle_X is injective. Since the space $HH(X)$ is finite dimensional this map is bijective. \square

Remark 3.6. *It follows from Lemma 3.4 and Corollary 3.5 that the Euler class*

$$Eu'(\Delta_* \mathcal{O}_X) \in HH(X) \otimes HH(X)$$

"is the pairing" \langle, \rangle_X . That is, if $\{e_i\}$ is a basis of $HH(X)$ and $\{f_i\}$ is the right-dual basis (i.e. $\langle e_i, f_j \rangle_X = \delta_{ij}$) then $Eu'(\Delta_ \mathcal{O}_X) = \sum f_i \otimes e_i$.*

Definition 3.7. *Let $E \in D^b(X \times X)$ and consider again the diagram*

$$X \times X \xleftarrow{\Delta} X \xrightarrow{p} \text{pt}.$$

Define the i -th Hochschild homology $HH_i(E)$ to be the space $\text{Hom}_{D^b(X)}^{-i}(\mathcal{O}_X, \mathbf{L}\Delta^ E)$. Then the total space $HH_\bullet(E)$ is finite dimensional because X is smooth and proper. Note that $HH(\Delta_* \mathcal{O}_X) = HH(X)$.*

Lemma 3.8. *Let $E \in D^b(X \times X)$ and let $Eu'(E) = \sum_s a_s \otimes b_s \in HH(X) \otimes HH(X)$. Then*

$$\sum_i (-1)^i \dim HH_i(E) = \sum_s \langle a_s, b_s \rangle_X$$

Proof. Properties 2,3,7 above imply that the following diagram commutes

$$\begin{array}{ccccccc}
D^b(\text{pt}) & \xrightarrow{\Phi_E} & D^b(X \times X) & \xrightarrow{\Delta^*} & D^b(X) & \xrightarrow{p_*} & D^b(\text{pt}) \\
\downarrow Eu & & \downarrow Eu & & \downarrow Eu & & \downarrow Eu \\
HH(\text{pt}) & \xrightarrow{HH(\Phi_E)} & HH(X \times X) & \xrightarrow{HH(\Delta^*)} & HH(X) & \xrightarrow{HH(p_*)} & HH(\text{pt})
\end{array}$$

Now the lemma immediately follows from Definitions 3.1, 3.7 and the last part in Property 7. \square

We are ready for the geometric Hochschild homology version of the Lefschetz fixed point theorem.

Theorem 3.9. *Let X be a smooth projective variety over a field k and $E \in D^b(X \times X)$. For each j consider the linear endomorphism $HH_j(\Phi_E)$ of $HH_j(X)$. Then there is the equality*

$$\sum_i (-1)^i \dim HH_i(E) = \sum_j (-1)^j \operatorname{Tr} HH_j(\Phi_E).$$

Proof. Choose a homogeneous basis $\{v_m\}$ of $HH_\bullet(X)$ and let $\{\bar{v}_m\} \subset HH_\bullet(X)$ be the left-dual basis with respect to \langle, \rangle_X , i.e. $\langle \bar{v}_m, v_n \rangle_X = \delta_{mn}$. Let

$$Eu(E)' = \sum_{m,n} \alpha_{mn} \cdot \bar{v}_m \otimes v_n$$

for $\alpha_{mn} \in k$.

Then by Lemma 3.8

$$\sum_i (-1)^i \dim HH_i(E) = \sum_m \alpha_{mm}.$$

On the other hand by Lemma 3.4

$$HH(\Phi_E)(v_l) = \sum_{m,n} \alpha_{mn} \cdot \langle v_l, \bar{v}_m \rangle_X \cdot v_n$$

By Property 5 above

$$\langle v_l, \bar{v}_m \rangle_X = (-1)^{\deg(v_l) \deg(\bar{v}_m)} \langle \bar{v}_m, v_l \rangle_X = (-1)^{\deg(v_l)} \langle \bar{v}_m, v_l \rangle_X = (-1)^{\deg(v_l)} \delta_{lm}.$$

So the trace of the linear operator $HH(\Phi_E)$ on $HH(X)$ equals $\sum_m (-1)^{\deg(v_m)} \alpha_{mm}$. And its supertrace is

$$\sum_j (-1)^j \operatorname{Tr} HH_j(\Phi_E) = \sum_m \alpha_{mm}$$

which proves the theorem. \square

Next we want to discuss the Hirzebruch-Riemann-Roch (HRR) theorem for Hochschild homology, which is closely related to the Lefschetz fixed point theorem.

Definition 3.10. *Let $E, F \in D^b(Y)$. We define the integer*

$$E \cdot F := \sum_j (-1)^j \dim H^j(Y, E \overset{\mathbf{L}}{\otimes} F).$$

It may be called the intersection of E and F .

Remark 3.11. If $Y = X \times X$ and F is the structure sheaf of the diagonal we have $E \cdot F = \sum_i (-1)^i HH_i(E)$. In particular $\Delta_* \mathcal{O}_X \cdot \Delta_* \mathcal{O}_X = \sum_i (-1)^i HH_i(X)$.

The next proposition is the HRR theorem for Hochschild homology.

Proposition 3.12. Let Y be a smooth projective variety and $E, F \in D^b(Y)$. Then

$$E \cdot F = \langle Eu(E), Eu(F) \rangle_Y.$$

Proof. The diagram

$$\begin{array}{ccccccc} D^b(\text{pt}) & \xrightarrow{\Phi_{E \boxtimes F}} & D^b(Y \times Y) & \xrightarrow{\Delta^*} & D^b(Y) & \xrightarrow{p_*} & D^b(\text{pt}) \\ \downarrow Eu & & \downarrow Eu & & \downarrow Eu & & \downarrow Eu \\ HH(\text{pt}) & \xrightarrow{HH(\Phi_{E \boxtimes F})} & HH(Y \times Y) & \xrightarrow{HH(\Delta^*)} & HH(Y) & \xrightarrow{HH(p_*)} & HH(\text{pt}) \end{array}$$

commutes by Property 7 above. By definition the number $E \cdot F$ is equal to the Euler characteristic of the complex of vector spaces $p_* \cdot \Delta^* \cdot \Phi_{E \boxtimes F}(k)$. Hence, by the last part of Property 7, it is equal to

$$E \cdot F = Eu \cdot p_* \cdot \Delta^* \cdot \Phi_{E \boxtimes F}(k).$$

On the other hand, by definition of the Euler class and Property 6 we have

$$HH(\Phi_{E \boxtimes F}) \cdot Eu(k) = K(Eu(E) \otimes Eu(F)).$$

It follows that

$$\langle Eu(E), Eu(F) \rangle_Y = HH(p_*) \cdot HH(\Delta^*) \cdot HH(\Phi_{E \boxtimes F}) \cdot Eu(k).$$

This proves the proposition. □

4. SINGULAR COHOMOLOGY OF SMOOTH COMPLEX PROJECTIVE VARIETIES

Assume now that $k = \mathbb{C}$.

Let X be a smooth complex projective variety and consider the singular cohomology $H^\bullet(X, \mathbb{C})$. It has the Hodge decomposition

$$H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^p(X, \Omega^q).$$

Let Y be another smooth complex projective variety and $f : X \rightarrow Y$ be a morphism. There is the induces degree preserving morphism on cohomology

$$f^* : H^\bullet(Y, \mathbb{C}) \rightarrow H^\bullet(X, \mathbb{C}),$$

and hence by Poincare duality $H^i(X, \mathbb{C})^* \simeq H^{2 \dim(X) - i}(X, \mathbb{C})$ and $H^i(Y, \mathbb{C})^* \simeq H^{2 \dim(Y) - i}(Y, \mathbb{C})$ the map

$$f_* : H^\bullet(X, \mathbb{C}) \rightarrow H^{\bullet + 2 \dim(Y) - 2 \dim(X)}(Y, \mathbb{C}).$$

Consider the projections $X \xleftarrow{q} X \times Y \xrightarrow{p} Y$. Then any class $\alpha \in H^\bullet(X \times Y)$ defines the corresponding convolution map

$$H^\bullet(X, \mathbb{C}) \rightarrow H^\bullet(Y, \mathbb{C}), \quad \beta \mapsto p_*(\alpha \cup q^*\beta).$$

For any object $S \in D^b(X)$ there is its Chern character

$$ch(S) \in \bigoplus_p H^p(X, \Omega_X^p) \subset H^\bullet(X, \mathbb{C}).$$

Recall also the Todd class $td_X \in \bigoplus_p H^p(X, \Omega_X^p)$ and its square root $\sqrt{td_X}$ (which is uniquely defined if one requires its degree zero term to be 1).

Definition 4.1. For any $S \in D^b(X)$ its Mukai vector

$$v(S) \in \bigoplus_p H^p(X, \Omega_X^p)$$

is the element $v(S) = ch(S) \cup \sqrt{td_X}$. This gives a map $v : D^b(X) \rightarrow H^\bullet(X, \mathbb{C})$ from objects of the derived category to the singular cohomology.

Definition 4.2. Any object $E \in D^b(X \times Y)$ defines the linear map $H^\bullet(\Phi_E) : H^\bullet(X, \mathbb{C}) \rightarrow H^\bullet(Y, \mathbb{C})$ which is the convolution with the Mukai vector $v(E)$

$$H^\bullet(\Phi_E)(\beta) = p_*(v(E) \cup q^*\beta).$$

It follows from the Grothendieck-Riemann-Roch theorem that the following diagram commutes

$$\begin{array}{ccc} D^b(X) & \xrightarrow{\Phi_E} & D^b(Y) \\ v \downarrow & & \downarrow v \\ H^\bullet(X, \mathbb{C}) & \xrightarrow{H^\bullet(\Phi_E)} & H^\bullet(Y, \mathbb{C}) \end{array}$$

The map $H^\bullet(\Phi_E)$ does not preserve the degree of the cohomology but it preserves the Hodge verticles

$$\bigoplus_{p-q=\text{fixed}} H^p(X, \Omega^q).$$

Hence $H^\bullet(\Phi_E)$ preserves the parity of the degree of the cohomology, i.e. it is the direct sum of operators $H^{\text{ev}}(\Phi_E)$ and $H^{\text{odd}}(\Phi_E)$.

Next is the singular cohomology version of LFP theorem for Fourier Mukai transforms.

Theorem 4.3. Let X be a smooth complex projective variety and let $E \in D^b(X \times X)$. Consider the induced linear operators

$$H^{\text{ev}}(\Phi_E) : H^{\text{ev}}(X, \mathbb{C}) \rightarrow H^{\text{ev}}(X, \mathbb{C}) \quad \text{and} \quad H^{\text{odd}}(\Phi_E) : H^{\text{odd}}(X, \mathbb{C}) \rightarrow H^{\text{odd}}(X, \mathbb{C}).$$

Then there is the equality

$$\sum_i (-1)^i \dim HH_i(E) = \mathrm{Tr} H^{\mathrm{ev}}(\Phi_E) - \mathrm{Tr} H^{\mathrm{odd}}(\Phi_E).$$

Proof. We deduce this theorem from Theorem 3.9.

Since X is a smooth variety over a field of characteristic zero one has the Hochschild-Kostant-Rosenberg isomorphism

$$I_{HKR}^X : HH_\bullet(X) \xrightarrow{\sim} \bigoplus_{p,q} H^p(X, \Omega_X^q),$$

which identifies the space $HH_i(X)$ with the Hodge vertical $\bigoplus_{p-q=-i} H^p(X, \Omega_X^q)$. Denote by I^X the composition

$$I^X : HH_\bullet(X) \xrightarrow{I_{HKR}^X} \bigoplus_{p,q} H^p(X, \Omega_X^q) \simeq H^\bullet(X, \mathbb{C}) \xrightarrow{\cup \sqrt{td_X}} H^\bullet(X, \mathbb{C})$$

We will use the following important result from [MaSte], Thm.1.2 (which in turn is heavily based on the work of Caldararu [Cal1],[Cal2] and Ramadoss [Ram]).

Theorem 4.4. *Let X and Y be smooth complex projective varieties and $\mathcal{E} \in D^b(X \times Y)$. Then the following diagram commutes*

$$\begin{array}{ccc} HH(X) & \xrightarrow{HH(\Phi_{\mathcal{E}})} & HH(Y) \\ I^X \downarrow & & \downarrow I^Y \\ H^\bullet(X, \mathbb{C}) & \xrightarrow{H^\bullet(\Phi_{\mathcal{E}})} & H^\bullet(Y, \mathbb{C}) \end{array}$$

We apply this theorem in case $X = Y$ and $\mathcal{E} = E$. Notice that I^X is an isomorphism, which preserves the parity of the cohomology space, i.e. it is an isomorphism of $\mathbb{Z}/(2)$ -graded spaces. This implies that the supertrace $\mathrm{Tr} H^{\mathrm{ev}}(\Phi_E) - \mathrm{Tr} H^{\mathrm{odd}}(\Phi_E)$ equals the supertrace $\sum_j (-1)^j \mathrm{Tr} HH_j(\Phi_E)$. So Theorem 4.3 follows from Theorem 3.9. \square

Remark 4.5. *Let Y be a smooth complex projective variety and let $E, F \in D^b(Y)$. The Hirzebruch-Riemann-Roch theorem implies that*

$$E \cdot F = \int_Y chE \cup chF \cup td_Y$$

(Definition 3.10). Let now $Y = X \times X$ and $F = \Delta_* \mathcal{O}_X$. By Grothendieck-Riemann-Roch theorem we have

$$ch(\Delta_* \mathcal{O}_X) = \Delta_*(td_X) \cup (td_{X \times X})^{-1}$$

Hence by Remark 3.11 and the above formula we get

$$\sum_i (-1)^i \dim HH_i(E) = \Delta_* \mathcal{O}_X \cdot E = \int_{X \times X} chE \cup \Delta_*(td_X) = \int_X \Delta^*(chE) \cup td_X.$$

This gives a formula for the left hand side in the LFP theorem in terms of the Chern character of E .

5. SINGULAR HOMOLOGY OF SMOOTH COMPLEX PROJECTIVE VARIETIES

Let X be a smooth complex projective variety and consider its singular homology

$$H_{\bullet}(X) = \bigoplus_j H_j(X, \mathbb{C}).$$

Let $f : X \rightarrow X$ be a morphism. Then one has the induced linear maps $H_j(f) : H_j(X, \mathbb{C}) \rightarrow H_j(X, \mathbb{C})$. Denote by $\Gamma(f) \subset X \times X$ the graph of the morphism f and consider its structure sheaf $\mathcal{O}_{\Gamma(f)}$ is an object in $D^b(X \times X)$. Next is the version of LFP theorem for singular homology.

Theorem 5.1. *In the previous notation there is the equality*

$$\sum_i (-1)^i \dim HH_i(\mathcal{O}_{\Gamma(f)}) = \sum_j (-1)^j \text{Tr } H_j(f).$$

Remark 5.2. *In case the graph $\Gamma(f)$ intersects the diagonal $\Delta \subset X \times X$ transversally (hence at a finite number of points) we have $\sum_i (-1)^i HH_i(\mathcal{O}_{\Gamma(f)}) = HH_0(\mathcal{O}_{\Gamma(f)})$ and $\dim HH_0(\mathcal{O}_{\Gamma(f)})$ is the number of fixed points of f . So one recovers the classical LFP theorem.*

Proof. We deduce Theorem 5.1 from Theorem 4.3 using Poincare duality.

Namely by Theorem 4.3 the number $\sum_i (-1)^i HH_i(\mathcal{O}_{\Gamma(f)})$ equals the supertrace of the linear operator $H^{\bullet}(\Phi_{\mathcal{O}_{\Gamma(f)}}) : H^{\bullet}(X, \mathbb{C}) \rightarrow H^{\bullet}(X, \mathbb{C})$. On the other hand using the Poincare duality $H^{2n-j}(X, \mathbb{C}) \simeq H_j(X, \mathbb{C})$ the map $H_{\bullet}(f)$ induces the map $f_* : H^{\bullet}(X, \mathbb{C}) \rightarrow H^{\bullet}(X, \mathbb{C})$. Notice that f_* preserves the degree of the cohomology, i.e. it is the sum of maps $f_*^s : H^s(X, \mathbb{C}) \rightarrow H^s(X, \mathbb{C})$. Thus it suffices to prove that

$$(5.1) \quad \sum_i (-1)^i HH_i(\mathcal{O}_{\Gamma(f)}) = \sum_s (-1)^s \text{Tr } f_*^s$$

So it remains to compare the linear maps $H^{\bullet}(\Phi_{\mathcal{O}_{\Gamma(f)}})$ and f_* and show that their supertraces are equal. This is achieved in the next lemma.

Lemma 5.3. *Let X and Y be complex projective varieties and $g : X \rightarrow Y$ be a morphism. Then the following diagram commutes*

$$\begin{array}{ccc} H^{\bullet}(X, \mathbb{C}) & \xrightarrow{H^{\bullet}(\Phi_{\mathcal{O}_{\Gamma(g)}})} & H^{\bullet}(Y, \mathbb{C}) \\ \downarrow \cup \sqrt{td_X} & & \downarrow \cup \sqrt{td_Y} \\ H^{\bullet}(X, \mathbb{C}) & \xrightarrow{g_*} & H^{\bullet}(Y, \mathbb{C}) \end{array}$$

The theorem follows from the lemma (applied to the case $Y = X$ and $g = f$) because the operator $\cup \sqrt{td_X}$ is an isomorphism which preserves the parity of the cohomology, so the supertraces of $H^\bullet(\Phi_{\mathcal{O}_{\Gamma(f)}})$ and f_* are equal. Thus it remains to prove the lemma.

Proof. Consider the diagram

$$X \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{q} \end{array} X \times Y \xrightarrow{p} Y$$

where p and q are the two projections and $i : X \rightarrow X \times Y$ is the isomorphism of X onto the graph $\Gamma(g)$ (so that $g = p \cdot i$). By definition

$$H^\bullet(\Phi_{\mathcal{O}_{\Gamma(g)}})(-) = p_*(ch(i_*\mathcal{O}_X) \cup \sqrt{td_{X \times Y}}) \cup q^*(-).$$

By Grothendieck-Riemann-Roch theorem

$$ch(i_*\mathcal{O}_X) \cup td_{X \times Y} = i_*(ch(\mathcal{O}_X) \cup td_X) = i_*(td_X)$$

hence

$$ch(i_*\mathcal{O}_X) = i_*(td_X) \cup (td_{X \times Y})^{-1}$$

and

$$\begin{aligned} H^\bullet(\Phi_{\mathcal{O}_{\Gamma(g)}})(-) &= p_*(i_*(td_X) \cup (\sqrt{td_{X \times Y}})^{-1} \cup q^*(-)) \\ &= p_*(i_*(td_X) \cup q^*(\sqrt{td_X})^{-1} \cup q^*(-) \cup p^*(\sqrt{td_Y})^{-1}) \\ &= p_*(i_*(td_X) \cup q^*((\sqrt{td_X})^{-1} \cup -)) \cup (\sqrt{td_Y})^{-1} \end{aligned}$$

Since $i^*q^* = \text{id}$ we have for any $\beta \in H^\bullet(X, \mathbb{C})$

$$i_*(td_X) \cup q^*\beta = i_*(td_X \cup i^*q^*\beta) = i_*(td_X \cup \beta)$$

Therefore

$$\begin{aligned} H^\bullet(\Phi_{\mathcal{O}_{\Gamma(g)}})(-) &= p_*(i_*(td_X \cup (\sqrt{td_X})^{-1} \cup -)) \cup (\sqrt{td_Y})^{-1} \\ &= p_*i_*(\sqrt{td_X} \cup -) \cup (\sqrt{td_Y})^{-1} \\ &= f_*(\sqrt{td_X} \cup -) \cup (\sqrt{td_Y})^{-1} \end{aligned}$$

This proves the lemma and Theorem 5.1 □

□

6. LEFSCHETZ FIXED POINT THEOREM FOR TWO MAPS

In this section we prove a generalization of Theorem 5.1 for two maps between different varieties of the same dimension. Namely, let X and Y be two smooth complex projective varieties and $f, g : X \rightarrow Y$ be morphisms. We obtain the induced maps

$$f_* : H_i(X) \rightarrow H_i(Y), \quad g^* : H^j(Y) \rightarrow H^j(X).$$

Assume now that $\dim X = \dim Y = d$. Then we get the diagram of maps

$$\begin{array}{ccc} H_i(X) & \xrightarrow{f_*} & H_i(Y) \\ D \uparrow & & \downarrow D \\ H^{2d-i}(X) & \xleftarrow{g^*} & H^{2d-i}(Y) \end{array}$$

where D denotes the Poincare duality isomorphisms on X and Y . We want a formula for the supertrace of the composition

$$D \cdot g^* \cdot D \cdot f_* : H_\bullet(X) \rightarrow H_\bullet(X).$$

Note that the composition $D \cdot f_* \cdot D : H^j(X) \rightarrow H^j(Y)$ is nothing but the push forward map f_* on cohomology which we considered in Section 5 above. Hence the supertrace of the composition $D \cdot g^* \cdot D \cdot f_*$ equals the supertrace of the composition

$$H^\bullet(X) \xrightarrow{f_*} H^\bullet(Y) \xrightarrow{g^*} H^\bullet(X)$$

We denote by $H^j(g^* \cdot f_*) : H^j(X) \rightarrow H^j(X)$ the restriction of this last composition to j -th cohomology.

Theorem 6.1. *Let X and Y be smooth complex projective varieties of the same dimension and let $f, g : X \rightarrow Y$ be two regular maps. Then in the previous notation there is the equality*

$$\mathcal{O}_{\Gamma(f)} \cdot \mathcal{O}_{\Gamma(g)} = \sum_j (-1)^j \operatorname{Tr} H^j(g^* \cdot f_*).$$

Hence the number $\mathcal{O}_{\Gamma(f)} \cdot \mathcal{O}_{\Gamma(g)}$ also equals the supertrace of the map $D \cdot g^* \cdot D \cdot f_*$ on homology of X .

Clearly Theorem 5.1 is a special case of Theorem 6.1 with $X = Y$ and $g = \operatorname{id}$.

Proof. We give the proof in two steps. In the first one we consider Hochschild homology and in the second - singular cohomology.

Step 1. Consider the graph $\Gamma(g) \subset X \times Y$ as a subvariety of $Y \times X$. We have the functors

$$(6.1) \quad \Phi_{\mathcal{O}_{\Gamma(f)}} : D^b(X) \rightarrow D^b(Y), \quad \Phi_{\mathcal{O}_{\Gamma(g)}} : D^b(Y) \rightarrow D^b(X)$$

which induce linear maps

$$HH(\Phi_{\mathcal{O}_{\Gamma(f)}}) : HH(X) \rightarrow HH(Y), \quad HH(\Phi_{\mathcal{O}_{\Gamma(g)}}) : HH(Y) \rightarrow HH(X)$$

By Property 3 in Section 3 their composition equals

$$(6.2) \quad HH(\Phi_{\mathcal{O}_{\Gamma(g)}}) \cdot HH(\Phi_{\mathcal{O}_{\Gamma(f)}}) = HH(\Phi_{\mathcal{O}_{\Gamma(f)} * \mathcal{O}_{\Gamma(g)}})$$

where $\mathcal{O}_{\Gamma(f)} * \mathcal{O}_{\Gamma(g)} \in D^b(X \times X)$ is the convolution of $\mathcal{O}_{\Gamma(f)}$ and $\mathcal{O}_{\Gamma(g)}$. By Theorem 3.9 and Remark 3.11 we have

$$(6.3) \quad (\mathcal{O}_{\Gamma(f)} * \mathcal{O}_{\Gamma(g)}) \cdot \Delta_* \mathcal{O}_X = \sum_j (-1)^j \operatorname{Tr} HH_j(\Phi_{\mathcal{O}_{\Gamma(f)} * \mathcal{O}_{\Gamma(g)}}).$$

Lemma 6.2. *Let $E, F \in D^b(X \times Y)$. We also consider F as an object in $D^b(Y \times X)$. Then*

$$(E * F) \cdot \Delta_* \mathcal{O}_X = E \cdot F.$$

*In particular there is the equality $(\mathcal{O}_{\Gamma(f)} * \mathcal{O}_{\Gamma(g)}) \cdot \Delta_* \mathcal{O}_X = \mathcal{O}_{\Gamma(f)} \cdot \mathcal{O}_{\Gamma(g)}$.*

Proof. Consider the obvious diagram

$$(6.4) \quad \begin{array}{ccccc} & & X \times Y \times Y \times X & & \\ & & \uparrow \Delta_Y & & \\ X \times Y \times X & \xrightarrow{p^Y} & X \times X & & \\ \uparrow \Delta_X & & \uparrow \Delta_X & & \\ X \times Y & \xrightarrow{p^Y} & X & \xrightarrow{p^X} & \text{pt} \end{array}$$

By definition the convolution $E * F \in D^b(X \times X)$ is the object $\mathbf{R}p_*^Y \cdot \mathbf{L}\Delta_Y^*(E \boxtimes F)$, and the number $(E * F) \cdot \Delta_* \mathcal{O}_X$ is the Euler characteristic of the complex

$$\mathbf{R}p_*^X \cdot \mathbf{L}\Delta_X^*(E * F) = \mathbf{R}p_*^X \cdot \mathbf{L}\Delta_X^* \cdot \mathbf{R}p_*^Y \cdot \mathbf{L}\Delta_Y^*(E \boxtimes F).$$

Since the square part of diagram 6.4 is cartesian, the map p^Y is smooth and all the varieties are smooth projective it follows from Lemma 1.3 in [BO] that there is a base change isomorphism of functors

$$\mathbf{L}\Delta_X^* \cdot \mathbf{R}p_*^Y \simeq \mathbf{R}p_*^Y \cdot \mathbf{L}\Delta_X^*.$$

Hence

$$\mathbf{R}p_*^X \cdot \mathbf{L}\Delta_X^*(E * F) = \mathbf{R}p_*^X \cdot \mathbf{R}p_*^Y \cdot \mathbf{L}\Delta_X^* \cdot \mathbf{L}\Delta_Y^*(E \boxtimes F).$$

Denote $\Delta := \Delta_Y \cdot \Delta_X : X \times Y \rightarrow X \times Y \times X \times Y$ (Δ is the diagonal embedding) and $p := p^X \cdot p^Y : X \times Y \rightarrow \text{pt}$. We have

$$\mathbf{R}p_*^X \cdot \mathbf{L}\Delta_X^*(E * F) = \mathbf{R}p_* \cdot \mathbf{L}\Delta^*(E \boxtimes F) = \mathbf{R}p_*(E \overset{\mathbf{L}}{\otimes} F)$$

and $E \cdot F$ is the Euler characteristic of the complex $\mathbf{R}p_*(E \overset{\mathbf{L}}{\otimes} F)$. This proves the lemma. \square

The lemma and the equality 6.3 imply that

$$(6.5) \quad \mathcal{O}_{\Gamma(f)} \cdot \mathcal{O}_{\Gamma(g)} = \sum_j (-1)^j \operatorname{Tr} HH_j(\Phi_{\mathcal{O}_{\Gamma(f)} * \mathcal{O}_{\Gamma(g)}}).$$

Step 2. The functors $\Phi_{\mathcal{O}_{\Gamma(f)}}, \Phi_{\mathcal{O}_{\Gamma(g)}}$ also induce the linear maps

$$H^\bullet(\Phi_{\mathcal{O}_{\Gamma(f)}}) : H^\bullet(X, \mathbb{C}) \rightarrow H^\bullet(Y, \mathbb{C}) \quad H^\bullet(\Phi_{\mathcal{O}_{\Gamma(g)}}) : H^\bullet(Y, \mathbb{C}) \rightarrow H^\bullet(X, \mathbb{C})$$

and by Theorem 4.4 the diagram

$$\begin{array}{ccccc} HH(X) & \xrightarrow{HH(\Phi_{\mathcal{O}_{\Gamma(f)}})} & HH(Y) & \xrightarrow{HH(\Phi_{\mathcal{O}_{\Gamma(g)}})} & HH(X) \\ \downarrow I^X & & \downarrow I^Y & & \downarrow I^X \\ H^\bullet(X, \mathbb{C}) & \xrightarrow{H^\bullet(\Phi_{\mathcal{O}_{\Gamma(f)}})} & H^\bullet(Y, \mathbb{C}) & \xrightarrow{H^\bullet(\Phi_{\mathcal{O}_{\Gamma(g)}})} & H^\bullet(X, \mathbb{C}) \end{array}$$

commutes. Since the map I^X is an isomorphism which preserves the parity of the grading it follows that the supertrace of the composition

$$(6.6) \quad H^\bullet(\Phi_{\mathcal{O}_{\Gamma(g)}}) \cdot H^\bullet(\Phi_{\mathcal{O}_{\Gamma(f)}}) : H^\bullet(X, \mathbb{C}) \rightarrow H^\bullet(X, \mathbb{C})$$

equals $\mathcal{O}_{\Gamma(f)} \cdot \mathcal{O}_{\Gamma(g)}$. So it suffices to prove the following proposition.

Proposition 6.3. *The supertrace of the composition 6.6 equals the supertrace*

$$\sum_j (-1)^j \text{Tr } H^j(g^* \cdot f_*).$$

Proof. By Lemma 5.3 above the following diagram commutes

$$\begin{array}{ccc} H^\bullet(X, \mathbb{C}) & \xrightarrow{H^\bullet(\Phi_{\mathcal{O}_{\Gamma(f)}})} & H^\bullet(Y, \mathbb{C}) \\ \downarrow \cup \sqrt{td_X} & & \downarrow \cup \sqrt{td_Y} \\ H^\bullet(X, \mathbb{C}) & \xrightarrow{f_*} & H^\bullet(Y, \mathbb{C}) \end{array}$$

The following lemma is similar.

Lemma 6.4. *Let X and Y be complex projective varieties and $g : X \rightarrow Y$ be a morphism. Then the following diagram commutes*

$$\begin{array}{ccc} H^\bullet(Y, \mathbb{C}) & \xrightarrow{H^\bullet(\Phi_{\mathcal{O}_{\Gamma(g)}})} & H^\bullet(X, \mathbb{C}) \\ \uparrow \cup \sqrt{td_Y} & & \uparrow \cup \sqrt{td_X} \\ H^\bullet(Y, \mathbb{C}) & \xrightarrow{g^*} & H^\bullet(X, \mathbb{C}) \end{array}$$

Proof. As in the proof of Lemma 5.3 we consider the diagram

$$X \xrightleftharpoons[q]{i} X \times Y \xrightarrow{p} Y$$

where p and q are the two projections and $i : X \rightarrow X \times Y$ is the isomorphism of X onto the graph $\Gamma(g)$ (so that $g = p \cdot i$).

By definition

$$H^\bullet(\Phi_{\mathcal{O}_{\Gamma(g)}})(-) = q_*(ch(i_* \mathcal{O}_X) \cup \sqrt{td_{X \times Y}}) \cup p^*(-).$$

By Grothendieck-Riemann-Roch theorem

$$ch(i_*\mathcal{O}_X) \cup td_{X \times Y} = i_*(ch(\mathcal{O}_X) \cup td_X) = i_*(td_X)$$

hence

$$ch(i_*\mathcal{O}_X) = i_*(td_X) \cup (td_{X \times Y})^{-1}$$

and

$$\begin{aligned} H^\bullet(\Phi_{\mathcal{O}_{\Gamma(g)}})(-) &= q_*(i_*(td_X) \cup (\sqrt{td_{X \times Y}})^{-1} \cup p^*(-)) \\ &= q_*(i_*(td_X) \cup q^*(\sqrt{td_X})^{-1} \cup p^*(-) \cup p^*(\sqrt{td_Y})^{-1}) \\ &= q_*(i_*(td_X) \cup q^*(\sqrt{td_X})^{-1} \cup p^*(- \cup (\sqrt{td_Y})^{-1})) \end{aligned}$$

Since $i^*q^* = \text{id}$ we have for any $\beta \in H^\bullet(X, \mathbb{C})$

$$i_*(td_X) \cup q^*\beta = i_*(td_X \cup i^*q^*\beta) = i_*(td_X \cup \beta)$$

Therefore

$$\begin{aligned} H^\bullet(\Phi_{\mathcal{O}_{\Gamma(g)}})(-) &= q_*(i_*(td_X \cup (\sqrt{td_X})^{-1}) \cup p^*(- \cup (\sqrt{td_Y})^{-1})) \\ &= q_*i_*(\sqrt{td_X} \cup i^*p^*(- \cup (\sqrt{td_Y})^{-1})) \\ &= \sqrt{td_X} \cup g^*(- \cup (\sqrt{td_Y})^{-1}) \end{aligned}$$

This proves the lemma. □

It follows from Lemmas 5.3 and 6.4 that

$$H^\bullet(\Phi_{\mathcal{O}_{\Gamma(g)}}) \cdot H^\bullet(\Phi_{\mathcal{O}_{\Gamma(f)}})(-) = \sqrt{td_X} \cup g^*(td_Y^{-1} \cup f_*(\sqrt{td_X} \cup -)).$$

Notice that the maps f_* and g^* preserve the degree of cohomology and the Todd class td is a power series in Chern classes with constant term 1. It follows that the supertrace (and the trace) of operators $H^\bullet(\Phi_{\mathcal{O}_{\Gamma(g)}}) \cdot H^\bullet(\Phi_{\mathcal{O}_{\Gamma(f)}})$ and $g^* \cdot f_*$ is the same. This proves the proposition and the theorem. □

□

Note that in the above proof of Theorem 6.1 the assumption $\dim X = \dim Y$ was used only at the very end when we derived Proposition 6.3 from Lemmas 5.3 and 6.4. (Without this assumption Theorem 6.1 is false: take for example $X = \mathbb{P}^1$ and $Y = \text{pt.}$)

In the above notation consider the composition of maps

$$H^\bullet(\Phi_{\mathcal{O}_{\Gamma(f)}}) \cdot H^\bullet(\Phi_{\mathcal{O}_{\Gamma(g)}}) = H^\bullet(\Phi_{\mathcal{O}_{\Gamma(f)} * \mathcal{O}_{\Gamma(g)}}).$$

This composition preserves the parity of cohomology and so is the sum of two maps $H^{\text{even}}(\Phi_{\mathcal{O}_{\Gamma(f)} * \mathcal{O}_{\Gamma(g)}})$ and $H^{\text{odd}}(\Phi_{\mathcal{O}_{\Gamma(f)} * \mathcal{O}_{\Gamma(g)}})$. Then our proof of Theorem 6.1 also gives the following

Theorem 6.5. *Let X and Y be smooth complex projective varieties and let $f, g : X \rightarrow Y$ be two regular maps. Then in the previous notation there is the equality*

$$\mathcal{O}_{\Gamma(f)} \cdot \mathcal{O}_{\Gamma(g)} = \mathrm{Tr} H^{\mathrm{even}}(\Phi_{\mathcal{O}_{\Gamma(f)} * \mathcal{O}_{\Gamma(g)}}) - \mathrm{Tr} H^{\mathrm{odd}}(\Phi_{\mathcal{O}_{\Gamma(f)} * \mathcal{O}_{\Gamma(g)}}).$$

Part 2. Lefschetz fixed point theorem for smooth and proper DG algebras

7. HOCHSCHILD HOMOLOGY OF DG CATEGORIES

Fix a ground field k . All algebras and categories are assumed to be k -linear. We write \otimes for \otimes_k unless specified otherwise. We follow consistently the universal sign rule: if x, y are homogeneous elements, then $xy = (-1)^{\deg(x)\deg(y)}yx$.

For a general discussion of DG algebras, DG modules, DG categories, etc. the reader may consult for example [BL],[Ke1],[Dr]. For us a *DG module* means a *right* DG module.

7.1. Hochschild homology. Let us recall the Hochschild complex and Hochschild homology of DG algebras and small DG categories [Ke2],[Shk].

Let $A = (A, d)$ be a DG algebra. As usual the suspension $sA = A[1]$ denotes the shift of grading: for $a \in A$ we have $\deg(sa) = \deg(a) - 1$. Consider the graded k -module

$$C_\bullet(A) = A \otimes T(A[1]) = \bigoplus_{n=0}^{\infty} A \otimes A[1]^{\otimes n}.$$

Its element $a_0 \otimes sa_1 \otimes sa_2 \otimes \dots \otimes sa_n$ is traditionally denoted by $a_0[a_1|a_2|\dots|a_n]$. The space $C_\bullet(A)$ is equipped with the differential $b = b_0 + b_1$, where b_0 and b_1 are anti-commuting differentials defined by

$$b_0(a_0) = da_0, \quad b_1(a_0) = 0,$$

and

$$\begin{aligned} b_0(a_0[a_1|\dots|a_n]) &= da_0[a_1|\dots|a_n] - \sum_{i=1}^n (-1)^{\eta_{i-1}} a_0[a_1|\dots|da_i|\dots|a_n], \\ b_1(a_0[a_1|\dots|a_n]) &= (-1)^{\deg(a_0)} a_0 a_1[a_2|\dots|a_n] + \sum_{i=1}^{n-1} (-1)^{\eta_i} a_0[a_1|\dots|a_i a_{i+1}|\dots|a_n] \\ &\quad - (-1)^{\eta_{n-1}(\deg(a_n)+1)} a_n a_0[a_1|\dots|a_{n-1}]. \end{aligned}$$

for $n \neq 0$, where $\eta_i = \deg(a_0) + \deg(sa_1) + \dots + \deg(sa_i)$. The complex $C_\bullet(A)$ is called the Hochschild chain complex of A , and the Hochschild homology is defined as

$$HH_n(A) = H^{-n}(C_\bullet(A)).$$

Similarly one defines the Hochschild chain complex $C_\bullet(\mathcal{A})$ and the Hochschild homology $HH(\mathcal{A}) = HH_\bullet(\mathcal{A})$ for a small DG category \mathcal{A} . Namely, denote by \mathcal{A}^{n+1} the set of sequences of objects $\{X_0, X_1, \dots, X_n\}$, $X_i \in \mathcal{A}$. For a fixed $\mathbb{X} = \{X_0, \dots, X_n\}$ denote by

$C_\bullet(\mathcal{A}, \mathbb{X})$ the graded space $\text{Hom}(X_n, X_0) \otimes \text{Hom}(X_{n_1}, X_n)[1] \otimes \dots \otimes \text{Hom}(X_0, X_1)[1]$. Now equip the space

$$C_\bullet(\mathcal{A}) = \bigoplus_{n \geq 0} \bigoplus_{\mathbb{X} \in \mathcal{A}^{n+1}} C_\bullet(\mathcal{A}, \mathbb{X})$$

with the differential $b = b_0 + b_1$ defined in analogy with the above case of a DG algebra. The complex $C_\bullet(\mathcal{A})$ is the Hochschild chain complex of the DG category \mathcal{A} and

$$HH_n(\mathcal{A}) = H^{-n}(C_\bullet(\mathcal{A}))$$

is the Hochschild homology of \mathcal{A} .

Clearly, a DG functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between DG categories \mathcal{A} and \mathcal{B} induces a morphism of complexes $C(F) : C_\bullet(\mathcal{A}) \rightarrow C_\bullet(\mathcal{B})$ and hence a morphism

$$HH(F) : HH(\mathcal{A}) \rightarrow HH(\mathcal{B}).$$

The following fact is proved in [Ke2].

Proposition 7.1. *Homotopy equivalent DG functors induce the same map on HH .*

Given a DG algebra A we denote by $A\text{-mod}$ the DG category of (right) DG A -modules. Let $\mathcal{P}(A) \subset A\text{-mod}$ be the full DG subcategory of h-projective DG modules. Then the (triangulated) homotopy category $Ho(\mathcal{P}(A))$ is equivalent to the derived category $D(A)$. Let $\text{Perf } A \subset \mathcal{P}(A)$ be the full DG subcategory of perfect DG modules, and $A^{\text{pre-tr}} \subset \text{Perf } A$ be the pre-triangulated envelop of the DG A -module A . Then by definition $Ho(\text{Perf } A)$ is the Karoubian closure of $Ho(A^{\text{pre-tr}})$. Here is another result from [Ke2].

Proposition 7.2. *The natural DG embeddings $A \rightarrow A^{\text{pre-tr}} \rightarrow \text{Perf } A$ induce isomorphisms*

$$HH(A) = HH(A^{\text{pre-tr}}) = HH(\text{Perf } A).$$

7.2. Kunneth isomorphism. Let A be a DG algebra. Let us recall the definition of the shuffle product

$$\text{sh} : C_\bullet(A) \otimes C_\bullet(A) \rightarrow C_\bullet(A).$$

For $a_0[a_1|\dots|a_n], b_0[b_1|\dots|b_m] \in C_\bullet(A)$ put

$$\text{sh}(a_0[a_1|\dots|a_n] \otimes b_0[b_1|\dots|b_m]) = (-1)^\heartsuit a_0 b_0 \text{sh}_{nm}[a_1|\dots|a_n|b_1|\dots|b_m]$$

Here $\heartsuit = \deg(b_0)(\deg(sa_1) + \dots + \deg(sa_n))$ and

$$\text{sh}_{nm}[x_1|\dots|x_n|x_{n+1}|\dots|x_{n+m}] = \sum_{\sigma} \pm [x_{\sigma^{-1}(1)}|\dots|x_{\sigma^{-1}(n)}|x_{\sigma^{-1}(n+1)}|\dots|x_{\sigma^{-1}(n+m)}]$$

where the sum is taken over all permutations that don't shuffle the first n and the last m elements and the sign is computed using the usual rule $xy = (-1)^{\deg(x)+\deg(y)}yx$.

Obviously, the shuffle product is functorial with respect to morphisms of DG algebras.

If B is another DG algebra, the natural homomorphisms of DG algebras $A \rightarrow A \otimes B$, $B \rightarrow A \otimes B$ induces morphisms of complexes $C_\bullet(A) \rightarrow C_\bullet(A \otimes B)$, $C_\bullet(B) \rightarrow C_\bullet(A \otimes B)$.

Theorem 7.3. *The composition K of maps*

$$C_\bullet(A) \otimes C_\bullet(B) \rightarrow C_\bullet(A \otimes B) \otimes C_\bullet(A \otimes B) \xrightarrow{\text{sh}} C_\bullet(A \otimes B)$$

is a morphism of complexes which is a quasi-isomorphism.

The Kunneth morphism K which is defined in the previous theorem for two DG algebras admits a generalization to the case of small DG categories [Shk],2.4, i.e. for DG categories \mathcal{A}, \mathcal{B} we get a functorial morphism

$$K : C_\bullet(\mathcal{A}) \otimes C_\bullet(\mathcal{B}) \rightarrow C_\bullet(\mathcal{A} \otimes \mathcal{B}).$$

Let A and B be DG algebras. The obvious DG functor

$$\text{Perf } A \otimes \text{Perf } B \rightarrow \text{Perf}(A \otimes B)$$

induces a morphism of complexes

$$C_\bullet(\text{Perf } A) \otimes C_\bullet(\text{Perf } B) \rightarrow C_\bullet(\text{Perf}(A \otimes B)).$$

We denote the composition

$$C_\bullet(\text{Perf } A) \otimes C_\bullet(\text{Perf } B) \xrightarrow{K} C_\bullet(\text{Perf } A \otimes \text{Perf } B) \rightarrow C_\bullet(\text{Perf}(A \otimes B))$$

again by K .

The next four lemmas are taken from [Shk],Prop.2.9,2.10,2.11,3.6.

Lemma 7.4. *This map K is a quasi-isomorphism.*

Proof. Indeed, consider the commutative diagram of complexes

$$\begin{array}{ccc} C_\bullet(A) \otimes C_\bullet(B) & \longrightarrow & C_\bullet(A \otimes B) \\ \downarrow & & \downarrow \\ C_\bullet(\text{Perf } A) \otimes C_\bullet(\text{Perf } B) & \longrightarrow & C_\bullet(\text{Perf}(A \otimes B)) \end{array}$$

The bottom arrow is a quasi-isomorphism because the other three are. □

Lemma 7.5. *Let A, B, C be DG algebras. The diagram*

$$\begin{array}{ccc} C_\bullet(\text{Perf } A) \otimes C_\bullet(\text{Perf } B) \otimes C_\bullet(\text{Perf } C) & \xrightarrow{K \otimes 1} & C_\bullet(\text{Perf}(A \otimes B)) \otimes C_\bullet(\text{Perf } C) \\ 1 \otimes K \downarrow & & \downarrow K \\ C_\bullet(\text{Perf } A) \otimes C_\bullet(\text{Perf}(B \otimes C)) & \xrightarrow{K} & C_\bullet(\text{Perf}(A \otimes B \otimes C)) \end{array}$$

commutes. That is, the map K is associative.

Let A, B, C, D be DG algebras. Let $X \in A^{\text{op}} \otimes C\text{-mod}$ and $Y \in B^{\text{op}} \otimes D\text{-mod}$ be bimodules which define functors

$$\Phi_X = - \otimes_A X : \text{Perf } A \rightarrow \text{Perf } C, \quad \Phi_Y = - \otimes_B Y : \text{Perf } B \rightarrow \text{Perf } D.$$

Lemma 7.6. *The following diagram*

$$\begin{array}{ccc} C_{\bullet}(\text{Perf } A) \otimes C_{\bullet}(\text{Perf } B) & \xrightarrow{K} & C_{\bullet}(\text{Perf}(A \otimes B)) \\ C(\Phi_X) \otimes C(\Phi_Y) \downarrow & & \downarrow C(\Phi_{X \otimes_k Y}) \\ C_{\bullet}(\text{Perf } C) \otimes C_{\bullet}(\text{Perf } D) & \xrightarrow{K} & C_{\bullet}(\text{Perf}(C \otimes D)) \end{array}$$

commutes.

Lemma 7.7. *For any DG algebra A the formula*

$$(a_0[a_1|\dots|a_n])^{\clubsuit} = (-1)^{n+\sum_{1 \leq i < j \leq n} \deg(sa_i) \deg(sa_j)} a_0[a_n|a_{n-1}|\dots|a_1]$$

defines a quasi-isomorphism $\clubsuit : C_{\bullet}(A) \rightarrow C_{\bullet}(A^{\text{op}})$. A similar formula defines a quasi-isomorphism $\clubsuit : C_{\bullet}(\mathcal{A}) \rightarrow C_{\bullet}(\mathcal{A}^{\text{op}})$ for any small DG category \mathcal{A} . Clearly this quasi-isomorphism is preserved by DG functors, i.e. given a DG functor $F : \mathcal{A} \rightarrow \mathcal{B}$ we have $\clubsuit \circ C(F) = C(F^{\text{op}}) \circ \clubsuit$. So we obtain a functorial isomorphism

$$\clubsuit : HH(\mathcal{A}) \xrightarrow{\sim} HH(\mathcal{A}^{\text{op}}).$$

7.3. Euler class. Let A be a DG algebra. Recall the definition of the Euler class map $\text{Perf } A \rightarrow HH_0(A)$. Given $N \in \text{Perf } A$ we consider the corresponding functor $\Phi_N = - \otimes_k N : \text{Perf } k \rightarrow \text{Perf } A$ and define

$$Eu(N) := HH(\Phi_N)(1) \in HH_0(\text{Perf } A) = HH_0(A).$$

Thus if B is another DG algebra and $F : \text{Perf } A \rightarrow \text{Perf } B$ is a DG functor then by definition $HH(F)(Eu(N)) = Eu(F(N))$.

The next two lemmas are Prop.3.1,3.2 in [Shk].

Lemma 7.8. *If $N, M \in \text{Perf } A$ are homotopy equivalent then $Eu(N) = Eu(M)$.*

Lemma 7.9. *For any exact triangle $L \rightarrow M \rightarrow N \rightarrow L[1]$ in $Ho(\text{Perf } A)$ we have*

$$Eu(M) = Eu(L) + Eu(N).$$

In particular $Eu(N[1]) = -Eu(N)$.

Corollary 7.10. *The map Eu descends to a group homomorphism*

$$Eu : K_0(Ho(\text{Perf } A)) \rightarrow HH_0(A)$$

Corollary 7.11. *Let $N \in \text{Perf } k$. Then $Eu(N) = \sum_i (-1)^i \dim H^i(N) \in k = HH_0(k) = HH_{\bullet}(k)$.*

7.4. Pairing on HH . Let A be a DG algebra. Consider A as a left DG A -bimodule, i.e. as a DG $A \otimes A^{\text{op}}$ -module via

$$(a \otimes b)c = (-1)^{\deg(b)\deg(c)}acb.$$

We denote by Δ this left DG A -bimodule.

Definition 7.12. Consider the DG functor $\Phi_{\Delta} = - \otimes_{A \otimes A^{\text{op}}} A : \text{Perf}(A \otimes A^{\text{op}}) \rightarrow \text{Perf}(k)$.

The composition of maps

$$C_{\bullet}(\text{Perf } A) \otimes C_{\bullet}(\text{Perf } A^{\text{op}}) \xrightarrow{K} C_{\bullet}(\text{Perf}(A \otimes A^{\text{op}})) \xrightarrow{C(\Phi_{\Delta})} C_{\bullet}(\text{Perf } k)$$

defines the pairing

$$\langle, \rangle = \langle, \rangle_A : HH(\text{Perf } A) \otimes HH(\text{Perf } A^{\text{op}}) \rightarrow k.$$

Using the canonical isomorphism $HH(A) = HH(\text{Perf } A)$ we also get the pairing

$$\langle, \rangle : HH(A) \otimes HH(A^{\text{op}}) \rightarrow k.$$

We can apply the previous construction to the DG algebra A^{op} instead of A to get the pairing

$$\langle, \rangle = \langle, \rangle_{A^{\text{op}}} : HH(A^{\text{op}}) \otimes HH(A) \rightarrow k.$$

Lemma 7.13. For $x \in HH(A), y \in HH(A^{\text{op}})$ we have

$$\langle x, y \rangle_A = (-1)^{\deg(x)\deg(y)} \langle y, x \rangle_{A^{\text{op}}}.$$

Proof. Denote by Δ^{op} the left DG $A^{\text{op}} \otimes A$ -module A via the action

$$(a \otimes b)c = (-1)^{\deg(a)(\deg(b)+\deg(c))}bca.$$

Then by definition the pairing $\langle, \rangle_{A^{\text{op}}}$ is defined by the composition of maps

$$C_{\bullet}(\text{Perf } A^{\text{op}}) \otimes C_{\bullet}(\text{Perf } A) \xrightarrow{K} C_{\bullet}(\text{Perf}(A^{\text{op}} \otimes A)) \xrightarrow{C(\Phi_{\Delta^{\text{op}}})} C_{\bullet}(\text{Perf } k)$$

Note that the isomorphism of DG algebras

$$\sigma : A \otimes A^{\text{op}} \rightarrow A^{\text{op}} \otimes A, \quad \sigma(a \otimes b) = (-1)^{\deg(a)\deg(b)}b \otimes a$$

interchanges the left DG modules Δ and Δ^{op} . Thus we obtain the induced commutative diagram of DG functors

$$\begin{array}{ccccc} A \otimes A^{\text{op}} & \longrightarrow & \text{Perf}(A \otimes A^{\text{op}}) & \xrightarrow{\Phi_{\Delta}} & \text{Perf } k \\ \sigma \downarrow & & \sigma \downarrow & & \parallel \\ A^{\text{op}} \otimes A & \longrightarrow & \text{Perf}(A^{\text{op}} \otimes A) & \xrightarrow{\Phi_{\Delta^{\text{op}}}} & \text{Perf } k \end{array}$$

Consider the diagram

$$\begin{array}{ccc} C_{\bullet}(A) \otimes C_{\bullet}(A^{\text{op}}) & \xrightarrow{K} & C_{\bullet}(A \otimes A^{\text{op}}) \\ & & \downarrow C_{\bullet}(\sigma) \\ C_{\bullet}(A^{\text{op}}) \otimes C_{\bullet}(A) & \xrightarrow{K} & C_{\bullet}(A^{\text{op}} \otimes A) \end{array}$$

It remains to notice that the induced isomorphism

$$K^{-1} \cdot HH(\sigma) \cdot K : HH(A) \otimes HH(A^{\text{op}}) \rightarrow HH(A^{\text{op}}) \otimes HH(A)$$

maps $x \otimes y$ to $(-1)^{\deg(x)\deg(y)} y \otimes x$. This implies the lemma. \square

Definition 7.14. Recall that for $M \in \text{Perf}(A \otimes A^{\text{op}})$ (resp. $M \in \text{Perf}(A^{\text{op}} \otimes A)$) the group $HH_i(M) := H^{-i}(\Phi_{\Delta}(M))$ (resp. $HH_i(M) := H^{-i}(\Phi_{\Delta^{\text{op}}}(M))$) is called the i -th Hochschild homology group of M .

7.5. Smooth and proper DG algebras. Recall that a DG algebra A is *smooth* if it is perfect as a DG $A^{\text{op}} \otimes A$ -module. It is called *proper* if its total cohomology $H(A)$ is finite dimensional.

Lemma 7.15. Let A and B be smooth and proper DG algebras.

- (a) The DG algebras A^{op} and $A \otimes B$ are also smooth and proper.
- (b) A DG A -module N is perfect if and only if its total cohomology $H(N)$ is finite dimensional.
- (c) Any DG module $L \in \text{Perf}(A^{\text{op}} \otimes B)$ defines the functor

$$\Phi_L : \text{Perf } A \rightarrow \text{Perf } B.$$

- (d) The total Hochschild homology $HH(A)$ is finite dimensional.
- (e) For $M \in \text{Perf}(A^{\text{op}} \otimes A)$ the total Hochschild homology $HH(M)$ is finite dimensional.

Proof. (a) See for example [Lu].

(b) Since A is proper it is clear that a perfect DG A -module has finite dimensional cohomology.

Vice versa, assume that N has finite dimensional cohomology. Since the DG algebra A is smooth, the DG $A^{\text{op}} \otimes A$ -module A is a homotopy direct summand of a DG $A^{\text{op}} \otimes A$ -module P which is obtained from $A^{\text{op}} \otimes A$ by an iterated cone construction. Then $N \simeq N \overset{\mathbf{L}}{\otimes}_A A$ is a homotopy direct summand of $N \otimes_A P$, which is obtained from the DG A -module $N \otimes_k A \simeq H(N) \otimes_k A$ by an iterated cone construction. Since $H(N)$ has finite dimension it follows that $N \in \text{Perf } A$.

(c) This follows from (b).

(d) Since $HH_{\bullet}(A) = H^{-\bullet}(A \overset{\mathbf{L}}{\otimes}_{A^{\text{op}} \otimes A} A)$ the statement is clear.

(e) This follows because $HH_{\bullet}(M) = H^{-\bullet}(M \overset{\mathbf{L}}{\otimes}_{A^{\text{op}} \otimes A} A)$. \square

8. MAIN THEOREMS

8.1. Formulation of main theorems.

Theorem 8.1. ([Shk], Thm.6.2) *Let A be a smooth and proper DG algebra. Then the pairing \langle, \rangle_A is nondegenerate.*

Theorem 8.2. (LFP) *Let A be a smooth and proper DG algebra and $M \in \text{Perf}(A^{\text{op}} \otimes A)$. Consider the functor $\Phi_M = - \otimes_A^{\mathbf{L}} M : \text{Perf } A \rightarrow \text{Perf } A$ and the corresponding linear endomorphisms $HH_j(\Phi_M) : HH_j(A) \rightarrow HH_j(A)$. Then there is an equality of the two elements of k*

$$\sum_i (-1)^i \dim HH_i(M) = \sum_j (-1)^j \text{Tr } HH_j(\Phi_M)$$

Theorem 8.3. (HRR [Shk], Thm.3.5) *Let A be a proper DG algebra. For any $N \in \text{Perf } A$, $M \in \text{Perf } A^{\text{op}}$*

$$\sum_i \dim H^i(N \otimes_A^{\mathbf{L}} M) = \langle Eu(N), Eu(M) \rangle.$$

Remark 8.4. *In the recent paper [Pe] there appears a generalization of Theorem 8.3 using the Euler class of a pair (M, f) , where $M \in \text{Perf } A$ and $f : M \rightarrow M$ is an endomorphism.*

8.2. Proofs of main Theorems. Everything is a consequence of the following key lemma proved in [Shk], Thm.3.4.

For $X \in \text{Perf}(A^{\text{op}} \otimes B)$ denote by $Eu'(X)$ the element

$$K^{-1}(Eu(X)) \in \bigoplus_n HH_{-n}(\text{Perf } A^{\text{op}}) \otimes HH_n(\text{Perf } B),$$

where K is the Kunneth isomorphism.

Note that if the DG algebra A is proper, then the functor $\Phi_X = - \otimes_A X$ maps $\text{Perf } A$ to $\text{Perf } B$.

Lemma 8.5. *Let A, B be DG algebras and $X \in \text{Perf}(A^{\text{op}} \otimes B)$. Assume that A is proper. Then the map*

$$HH(\Phi_X) : HH(A) \rightarrow HH(B)$$

is the convolution with the class $Eu'(X)$. That is, if

$$Eu'(X) = \sum_n x'_{-n} \otimes x_n \in \bigoplus_n HH_{-n}(\text{Perf } A^{\text{op}}) \otimes HH_n(\text{Perf } B),$$

then $HH(T_X)(y) = \sum_n \langle y, x'_{-n} \rangle \cdot x_n$.

Proof. Note that the DG functor Φ_X is isomorphic to the composition of DG functors

$$\mathrm{Perf} A \xrightarrow{-\otimes_k X} \mathrm{Perf}(A \otimes A^{\mathrm{op}} \otimes B) \xrightarrow{\Phi_{\Delta \otimes_k B}} \mathrm{Perf} B$$

It follows that the corresponding map $HH(\Phi_X)$ is isomorphic to the following composition

$$\begin{array}{ccc} HH(\mathrm{Perf} A) \otimes HH(\mathrm{Perf} k) & & HH(\mathrm{Perf} B) \\ \downarrow & & \uparrow \\ HH(\mathrm{Perf} A) & & HH(\mathrm{Perf} k) \otimes HH(\mathrm{Perf} B) \\ HH(-\otimes_k X) \downarrow & & \uparrow HH(\Phi_{\Delta}) \otimes \mathrm{id} \\ HH(\mathrm{Perf}(A \otimes A^{\mathrm{op}} \otimes B)) & & HH(\mathrm{Perf}(A \otimes A^{\mathrm{op}})) \otimes HH(\mathrm{Perf} B) \\ K^{-1} \downarrow & & \uparrow K \otimes \mathrm{id} \\ HH(\mathrm{Perf} A) \otimes HH(A^{\mathrm{op}} \otimes B) & \xrightarrow{\mathrm{id} \otimes K^{-1}} & HH(\mathrm{Perf} A) \otimes HH(\mathrm{Perf} A^{\mathrm{op}}) \otimes HH(\mathrm{Perf} B) \end{array}$$

The composition of the left vertical arrows equals

$$HH(\mathrm{Perf} A) \otimes HH(\mathrm{Perf} k) \xrightarrow{\mathrm{id} \otimes Eu'(X)} HH(\mathrm{Perf} A) \otimes HH(\mathrm{Perf}(A^{\mathrm{op}} \otimes B)).$$

This implies the lemma. \square

8.3. Proof of Theorem 8.3. We apply Lemma 8.5 with $A = A, B = k$ and $X = M$. The composition of DG functors

$$\mathrm{Perf} k \xrightarrow{\Phi_N} \mathrm{Perf} A \xrightarrow{\Phi_M} \mathrm{Perf} k$$

is isomorphic to the DG functor $\Phi_{N \otimes_A M} : \mathrm{Perf} k \rightarrow \mathrm{Perf} k$. By Lemma 8.5

$$Eu(N \otimes_A M) = \langle Eu(N), Eu(M) \rangle$$

and by Corollary 7.11 $Eu(N \otimes_A M) = \sum_i (-1)^i \dim H^i(N \otimes_A M)$.

8.4. Proof of Theorem 8.1. We apply Lemma 8.5 with $A = B = X$. Then the functor $\Phi_X : \mathrm{Perf} A \rightarrow \mathrm{Perf} A$ is isomorphic to the identity. Therefore the corresponding linear map $HH(\Phi_X) : HH(A) \rightarrow HH(A)$ is the identity. By Lemma 8.5 this shows that the map $HH(A) \rightarrow HH(A^{\mathrm{op}})^*$ defined by the pairing \langle, \rangle_A is injective. Since the space $HH(A)$ is finite dimensional and is isomorphic to $HH(A^{\mathrm{op}})$ it follows that the pairing is nondegenerate.

8.5. Proof of Theorem 8.2. Fix $M \in \mathrm{Perf}(A^{\mathrm{op}} \otimes A)$. As before denote by $Eu(M)' \in HH(A^{\mathrm{op}}) \otimes HH(A)$ the inverse image of $Eu(M)$ under the Kunnet isomorphism

$$HH(A^{\mathrm{op}}) \otimes HH(A) \xrightarrow{K} HH(A^{\mathrm{op}} \otimes A).$$

Choose a homogeneous basis $\{v_m\}$ of $HH(A)$ and let $\{\bar{v}_m\}$ be a basis of $HH(A^{\text{op}})$ such that $\langle \bar{v}_m, v_n \rangle_{A^{\text{op}}} = \delta_{mn}$ (we use Theorem 8.1). Let

$$Eu(M)' = \sum_{m,n} \alpha_{mn} \cdot \bar{v}_m \otimes v_n$$

for $\alpha_{mn} \in k$. Then by Definitions 7.12, 7.14 and Corollary 7.11

$$\sum_i (-1)^i \dim HH_i(M) = \sum_m \alpha_{mm}.$$

On the other hand by Lemma 8.5

$$HH(\Phi_M)(v_l) = \sum_{m,n} \alpha_{mn} \cdot \langle v_l, \bar{v}_m \rangle_A \cdot v_n$$

By Lemma 7.13 $\langle v_l, \bar{v}_m \rangle_A = (-1)^{\deg(v_l) \deg(\bar{v}_m)} \langle \bar{v}_m, v_l \rangle_{A^{\text{op}}} = (-1)^{\deg(v_l)} \langle \bar{v}_m, v_l \rangle_{A^{\text{op}}} = (-1)^{\deg(v_l)} \delta_{lm}$. So the trace of the linear operator $HH(\Phi_M)$ on $HH(A)$ equals $\sum_m (-1)^{\deg(v_m)} \alpha_{mm}$.

Hence its supertrace is

$$\sum_i (-1)^i \text{Tr } HH_i(\Phi_M) = \sum_m \alpha_{mm}$$

which proves the theorem.

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